This question concerns a simple model for the formation of rings around a planet. The system we consider is a central planet of mass $M$ around which a cloud of small particles of identical mass $m << M$ revolve. There is no particular symmetry in the cloud initially; however, the total angular momentum of the cloud around the planet is non-zero. Assume that the particles do not interact with each other except for collisions that obey the following simplified collision rule:

$$v_i' - v_j' = -e_r (v_i - v_j),$$

where $e_r$ (the coefficient of restitution) obeys $0 < e_r < 1$, $v_i$ is the velocity of the $i$-th particle, and primes denote the velocities after collision. The collisions are instantaneous and do not change the masses of the colliding particles.

(a) Consider two colliding particles with velocities before collision $v_1$ and $v_2$. Write down the loss of kinetic energy $\Delta K$ in terms of $m$, $e_r$ and the velocities before the collision.

(b) Suppose that the particles in (a) have angular momentum $L_1$ and $L_2$ around the central planet. Demonstrate that upon collision, $|L_1 - L_2|$ diminishes.

(c) The Laplace-Runge-Lenz vector of a particle is defined by

$$\varepsilon = \frac{\mathbf{v} \times \mathbf{L}}{Gm(M+m)} - \frac{\mathbf{r}}{r},$$

where $\mathbf{L}$ is the particle angular momentum, $\mathbf{v}$ the particle velocity, $\mathbf{r}$ the particle position whose origin is at the central planet and $G$ the gravitational constant. Demonstrate that upon collision the magnitude of the difference between the Laplace-Runge-Lenz vectors of the colliding particles diminishes.

(d) Given that $\mathbf{L}$ and $\varepsilon$ uniquely determine the Kepler orbit of the particle, and that the Laplace-Runge-Lenz vector is invariant when there are no collisions, use your results from (b) and (c) to argue that eventually stable rings are formed around the planet.