Consider a particle of mass $m$ moving in a plane under the influence of a velocity-dependent potential. The system has Lagrangian

$$\mathcal{L}(\mathbf{r}, \mathbf{v}) = \frac{1}{2} m |\mathbf{v}|^2 - V_0 \frac{1 - \alpha \dot{r}^2}{r}$$

where $r = |\mathbf{r}|$ is the distance from the origin, $\mathbf{v} = \frac{d}{dt} \mathbf{r}$ is the velocity, $V_0$ and $\alpha$ are positive constants, and the dot denotes a time derivative.

a) Write the Lagrangian in polar coordinates $(r, \theta)$. From it find the corresponding canonical momenta $p_r$, $p_\theta$, and also the Hamiltonian.

b) Show from the Hamiltonian equations of motion that the angular momentum $p_\theta$ and energy $E$ are conserved.

c) Reduce the equations of motion to a single first order differential equation. (It is not necessary to solve this equation further.)

d) For a solution with a given positive energy $E$ and angular momentum $p_\theta$, what is the distance of closest approach to the origin? Why does your answer not depend on the value of $\alpha$?