The Hamiltonian of a harmonic oscillator of mass $m$ in one dimension is

$$\hat{H} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 \hat{x}^2$$

where the frequency, $\omega$, is a constant. The energy of the $n^{th}$ state, $|n\rangle$, is

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega .$$

Raising and lowering operators

$$\hat{a}_+ = \frac{1}{\sqrt{2m}}(\hat{p} \pm i\hbar \omega \hat{x})$$

may be defined, which have the properties

$$\hat{a}_+ |n\rangle = i\sqrt{(n+1)\hbar\omega} |n+1\rangle$$
$$\hat{a}_- |n\rangle = -i\sqrt{n\hbar\omega} |n-1\rangle .$$

a) Calculate the commutator $[\hat{a}_+, \hat{a}_-]$.

b) Consider adding the interaction

$$\hat{H}' = b\hat{x}$$

(where $b$ is a constant) to the Hamiltonian. Show that, at first order in perturbation theory, the energy of the $n^{th}$ state does not change. You may use raising and lowering operators, but are not required to do so.

c) The relativistic correction to the kinetic energy is

$$\hat{H}' = -\frac{\hat{p}^4}{8m^3c^2} .$$

Use perturbation theory to calculate the first-order correction to the harmonic oscillator energies. You may use raising and lowering operators, but are not required to do so.