A particle of mass $m$ moves in a one-dimensional potential well having both infinitely high potential walls at $x = \pm a$, and an attractive one-dimensional $\delta$-function potential well of strength $\alpha$ located at $x = 0$:

$$V(x) = -\alpha \delta(x) \quad |x| < a$$
$$V(x) = \infty \quad |x| \geq a$$

(a) Show by integrating the Schrödinger equation across the $\delta$-function that the wavefunction for this potential exhibits a discontinuous change of slope across the $\delta$-function given by:

$$\psi(x) = \frac{1}{\sqrt{2m\alpha}} \exp\left(-\frac{x}{\sqrt{2m\alpha}}\right)$$

(b) Assuming that $\alpha$ is small, qualitatively sketch the wavefunctions associated with (i) the ground state, (ii) the 1$^{\text{st}}$ excited state, and (iii) the 2$^{\text{nd}}$ excited state of the particle confined in the potential above.

(c) Assume that $\alpha$ is small, so that the $\delta$-function potential can be treated as a small perturbation on the infinite square well potential ($\alpha = 0$). For the lowest three energy levels, use first-order perturbation theory to estimate the energy difference $\Delta E_n^{(1)}$ between the eigenvalues of the full potential shown above ($\alpha \neq 0$), and those of the infinite square well potential ($\alpha = 0$).
(d) If the infinite potential walls are moved to $x = \pm \infty$ (as shown below), for finite $\alpha$, one can show that there will be one bound state of the $\delta$-function potential with an energy $E_1$. Sketch this bound state wavefunction of the $\delta$-function potential.

(e) Use the variational trial wavefunction $\psi_{tr} = Ae^{-bx^2}$ to obtain an upper-bound on the bound state energy of the $\delta$-function potential $E_1$.

The following integrals may be useful:

$$\int_{-\infty}^{\infty} x^n e^{-ax^2} dx = \frac{1 \cdot 3 \cdot 5 \cdots}{2^n a^n} \sqrt{\frac{\pi}{a}}$$

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}.$$