Consider a one-dimensional, symmetric double potential well system (well height $V_0$; well boundaries at $x = \pm a, \pm b$, as illustrated below), that contains a quantum mechanical particle of energy $E$. Assume $E < V_0$.

**(a)** Write down general solutions to the Schrödinger equation in regions I, II, III, IV, and V in terms of amplitudes, $E$, $V_0$, $x$, and fundamental constants. State the boundary conditions necessary for determining the amplitudes. Do not solve for the amplitudes.

**(b)** When the potential wells are well-separated so that we can make a two-state approximation, the basis states $\psi_L$ and $\psi_R$ are the groundstate wavefunctions confined to the left and right wells, respectively. In addition, each groundstate has energy $E_0$.

(i) Sketch the groundstate wavefunctions (first copy the illustration above into your exam books, then make the sketch).

(ii) If the basis states have an interaction energy $\Delta E$, express the Hamiltonian for this "tunneling" system as a $2 \times 2$ matrix. Calculate the eigenvalues and eigenvectors for this system.

**(c)** Sketch the two lowest energy eigenstates of the tunneling system.

**(d)** Consider the case where the energies of $\psi_L$ and $\psi_R$ are changed by a small amount, $\alpha$ and $-\alpha$, respectively. The quantity $\alpha \ll V_0$, and the basis states still have interaction energy $\Delta E$.

(i) Express the Hamiltonian for this system as a $2 \times 2$ matrix.

(ii) Sketch the two lowest energy eigenstates of this system. Is the energy difference between the two lowest energy states bigger, smaller, or the same as the case where $\alpha = 0$?

**(e)** Going back to the case of unperturbed ($\alpha = 0$) basis states, let's now consider the time-dependence of the system. Let the particle initially be in the left well, so that $\psi(t=0) = \psi_L$. Suppose that the particle can tunnel through the potential barrier in a time $\tau$. Estimate $\tau$ in terms of $\Delta E$.

**(f)** Now explicitly calculate the time-dependence of $\psi_L(t)$ to show that the particle eventually returns to its initial state (at least, to within a phase factor).