(a) A beam of particles of mass $m_b$ and momentum $p_i$ is scattered by a weak potential $H_I$ that can be considered as a small perturbation: $H = H_0 + H_I$,

$$H_0 = -\frac{\hbar^2}{2m_b} \nabla_b^2 \quad \text{and} \quad H_I = \lambda e^{-(r_b/\alpha)^2}$$

where $r_b$ is the position of the particle measured from the scattering center.

(a.1) What is the magnitude $|q|$ of the momentum transferred by the potential to the scattered particle as a function of $|p_i|$ and the scattering angle $\theta$.

(a.2) Using the golden rule for transition probability:

$$T_{i\rightarrow f} = \frac{2\pi}{\hbar} |\langle f | H_I | i \rangle|^2 \rho_f,$$

where $\rho_f$ is the density of final states, calculate the differential scattering cross section:

$$d\sigma = \frac{T_{i\rightarrow f}}{\text{incident flux}}$$

to order $\lambda^2$. Your answer may contain $|q|$.
(b) Now consider the case in which the above beam of particles is incident on a target particle of mass \( m_t \) bound in a potential \( W(r_t) \). The Hamiltonian is

\[
H_0 = -\frac{\hbar^2}{2m_b} \nabla_b^2 + H_t \quad \text{where} \quad H_t = -\frac{\hbar^2}{2m_t} \nabla_t^2 + W(r_t)
\]

and

\[
H_1 = \lambda e^{-(r_b-r_t)^2/\alpha^2}
\]

Let \( \phi_0(r_t) \) and \( \phi_1(r_t) \) be the normalized ground and first excited eigenfunctions of \( H_t \) with energies \( E_0 \) and \( E_1 \). We observe both elastic scattering, and also inelastic scattering where the target is excited to the state \( \phi_1(r_t) \).

(b.1) What are the wave functions of the initial and final states for elastic and inelastic scattering. You may use \( p_f \) to denote the momentum of the scattered particle without calculating it as a function of \( \theta \).

(b.2) Calculate the elastic and inelastic scattering cross sections to order \( \lambda^2 \). You may use \( q \) to denote momentum transfer without calculating it as a function of \( \theta \).

(b.3) Show that in the limit \( |q| \to 0 \) the elastic scattering cross section becomes identical to that for scattering by the potential \( \lambda e^{-(r_b/\alpha)^2} \) considered in part (a), while the inelastic cross section becomes zero.

Note: \[
\int d^3r \ e^{-r^2/\alpha^2} e^{-i\mathbf{k} \cdot \mathbf{r}} = \pi^{3/2} \alpha^3 e^{-k^2 \alpha^2/4}.
\]