A cesium atom used in an atomic clock is driven by a classical, oscillating magnetic field between two hyperfine states $|a\rangle$ and $|b\rangle$. The Hamiltonian which describes the atom interacting with the classical field is:

$$H = \frac{\hbar \omega_0}{2} (|a\rangle\langle a| - |b\rangle\langle b|) + \hbar 2\Omega \cos(\omega_0 t)(|a\rangle\langle b| + |b\rangle\langle a|)$$

where the “coupling” $\Omega$ is a real number.

(a) One can write the atomic wavefunction as

$$\psi(t) = c_a(t)|a\rangle + c_b(t)|b\rangle.$$ 

Using Schrödinger’s equation, write down differential equations for the coefficients $c_a(t)$ and $c_b(t)$.

(b) At the beginning of an experiment, the atom is prepared in state $|a\rangle$ at time $t=0$.

Defining the coefficients $c_a' = e^{\frac{i \omega_0 t}{2}} c_a$ and $c_b' = e^{-\frac{i \omega_0 t}{2}} c_b$, show that the coefficients $c_a'$ and $c_b'$ satisfy the simultaneous differential equations

$$i c_a' = \Omega c_b'$$
$$i c_b' = \Omega c_a'$$

Work in the $\Omega \ll \omega_0$ limit and ignore oscillations in the amplitudes $c_a(t)$ and $c_b(t)$ at frequencies much higher than $\Omega$. Hence calculate $c_a(t)$ and $c_b(t)$.

(c) After how much time is the atom 100% likely to be measured in state $|b\rangle$?

(d) If the phase of the classical field is changed so that the Hamiltonian is

$$H = \frac{\hbar \omega_0}{2} (|a\rangle\langle a| - |b\rangle\langle b|) + \hbar 2\Omega \cos(\omega_0 t + \phi)(|a\rangle\langle b| + |b\rangle\langle a|)$$

and the experiment is repeated, does your answer to part (c) change? Why or why not?