A molecule consists of three equal atoms which form an equilateral triangle of side \( L \) as shown. The eigenstates of an electron in this molecule can be considered as linear combinations of orthonormal basis states, \( \phi_i \), centered on each atom \( i = 1,2,3 \). In this basis the Hamiltonian has diagonal matrix elements \( \langle \phi_i | H | \phi_i \rangle = \varepsilon \), and off-diagonal matrix elements between neighboring basis states \( i \) and \( j \), \( \langle \phi_i | H | \phi_j \rangle = -t \), where the energy \( t > 0 \).

(a) Define the operator \( R \) which operates on the basis functions as follows:
\[
R \phi_i = \phi_{i+1}, \quad i = 1,2; \quad R \phi_3 = \phi_1.
\]
Note that \( R \) has the effect of rotating the basis functions around the triangle and that \( R^3 = 1 \). Show that \( R \) commutes with the Hamiltonian.

(b) The eigenstates \( \psi_n \) of \( R \) with eigenvalues \( \lambda_n \) can be expressed as
\[
\psi_n = \sum_{i=1}^{3} a_{ni} \phi_i.
\]
Find the expansion coefficients \( a_{ni} \) and the eigenvalues \( \lambda_n \).

(Hint: note that \( \lambda_n = 1 \).)

(c) Give the energy eigenvalues in terms of \( \varepsilon \) and \( t \) and give their degeneracies.

(d) Now suppose an electric field of strength \( F \) is switched on in the +x-direction. The only effect is to change the diagonal matrix elements of the Hamiltonian at each site by an amount \( eFx_i \), where \( x_i \) is the x position of the center of atom \( i \), and \( -e \) is the charge of the electron. Find the new energy eigenvalues to linear order in \( F \).