Consider the quantum mechanics of an electron with charge $-e$ and mass $M$ moving on a ring of radius $R$ located in the $xy$ plane. The electron moves in the presence of a uniform magnetic field $\mathbf{B} = B\hat{z}$. The quantum mechanical Hamiltonian of this system is

$$H = \frac{1}{2MR^2} \left( p_\varphi - \frac{e\Phi}{2\pi c} \right)^2$$

where $\varphi$ is the angular coordinate of the particle on the ring, $p_\varphi = \frac{\hbar}{i} \frac{\partial}{\partial \varphi}$ is the momentum conjugate to $\varphi$, $\Phi = B\pi R^2$ is the magnetic flux and $c$ is the speed of light.

(a) What commutation relations do the operators $\varphi$ and $p_\varphi$ obey? What boundary conditions do the wave functions of this system satisfy? Justify your answer.

(b) Find the exact eigenstates and the energy levels $E_n$ of this system, giving the physical meaning and the allowed values of the quantum numbers which label the states. Write your answers in terms of the magnetic flux $\Phi$ and the flux quantum $\phi_0 = 2\pi \frac{\hbar c}{e}$.

Hint: Write the flux $\Phi$ as $\Phi = (k + \alpha)\phi_0$, where $k$ is an integer and $\alpha$ is a real number in the interval $-\frac{1}{2} \leq \alpha < \frac{1}{2}$.

(c) Find the ground state of this system for arbitrary values of the magnetic flux $\Phi$. For what values of $\Phi$ does the ground state become degenerate? What are the quantum numbers of the degenerate states? Sketch the energy of the ground state as a function of $\Phi / \phi_0$. 