Consider three interacting spins arranged in an equilateral triangle as shown. The $z$ component of each spin $s_i \ (i = 1, 2, 3)$ can take values $s_i = \pm 1$. There is a magnetic field $B$ in the $z$ direction acting on each spin so that the Hamiltonian is given by

$$H = -J(s_1s_2 + s_1s_3 + s_2s_3) + \mu_B B \sum_{i=1}^{3} s_i,$$

where $\mu_B$ is the Bohr magneton and $J$ is the interaction constant.

(a) Give explicit expressions for:

1. the partition function at temperature $T$
2. the expectation value $\langle S \rangle$ of the total spin $S = \sum_{i=1}^{3} s_i$
3. the expectation value of the term in the energy involving $J$.

(b) Give the expected form of the spin susceptibility $d\langle S \rangle/dB$ as a function of temperature $T$ for large $T$.

(c) Show that the result in (b) follows from the expression for $\langle S \rangle$ derived in part (a), by formal calculation.

(d) Copy the figure shown below into your exam book. Plot on this graph in your exam book the zero-temperature phase diagram of the spin system as a function of $J$ and $B$. Note that the diagram includes both $J > 0$ and $J < 0$. Sketch the configurations of the spins in each region of the phase diagram.