N identical classical particles occupy a square lattice with 2N sites, with at most one particle per site. Alternate sites are labeled A and B, as sketched below.

Denote by $c$ the fraction of particles on the A sites.

A) For fixed $c$, and assuming that all configurations at fixed $c$ are equally likely, (a mean field approximation) calculate the entropy $S(c)$ of the system for large values of $N$. Evaluate the entropy when $c = \frac{1}{2}$.

B) When two objects are on neighboring A and B sites, there is a repulsive interaction energy $E_0$. For fixed $c$, and assuming that all configurations at fixed $c$ are equally likely, show that the average total energy of the system is

$$E(c) = 4NE_0 c(1-c).$$

In thermal equilibrium at temperature $T$, $c$ is determined by minimizing the free energy $F(c) = E(c) - TS(c)$. This system exhibits a second order phase transition at a temperature $T_c$.

C) Describe the state of the system at very high temperatures. What is the observed value of $c$?

D) Describe the state of the system at very low temperatures. What are the possible values of $c$?

E) Determine $T_c$.

HINT: For large values of $N$ we can approximate $\ln N!$ by $N\ln N - N$.

For small $x$, $\ln(1 + x) = x - x^2/2 + x^3/3 + \ldots$. 