Consider the quantum mechanics of a particle of mass $m$ in a 3-dimensional isotropic harmonic oscillator potential and governed by the Hamiltonian

$$H_0 = \frac{1}{2m} \left( p_x^2 + p_y^2 + p_z^2 \right) + \frac{1}{2} m \omega^2 \left( x^2 + y^2 + z^2 \right),$$

where $\omega$ is a positive constant.

a) Write down (no calculation required) the energies of the three lowest distinct energy levels of $H_0$ and state their degeneracy.

b) Suppose this system is perturbed by the potential

$$V = \lambda m \omega xy,$$

where $\lambda > 0$.

Find the physical constraint on $\lambda$ required for the energy to be bounded from below.

c) Show that a suitable rotation of the axes reduces the problem to that of three uncoupled oscillators and hence write down an exact expression for the energy spectrum of $H = H_0 + V$.

d) Assuming $0 < \lambda \ll 1$, use perturbation theory to find the shift in the ground state energy to first order in $\lambda$.

e) Take the limit of your answer from part (c) as $\lambda \to 0$ and compare your answer with your perturbation calculation from part (d). Do they agree? If not why not?

**Hint:** You may use without proof the usual relations between position/momentum and creation/annihilation operators

$$x = \sqrt{\frac{\hbar}{2m\omega}} \left( a_x + a_x^\dagger \right) \text{ and } p_x = i \sqrt{\frac{m\hbar\omega}{2}} \left( -a_x + a_x^\dagger \right),$$

and similarly for $y$, $p_y$ and $z$, $p_z$. 
